LEMMA: *If a natural number N can be written as the sum of an even number and $3 * b$, where $b \geq 0$, then either N is even or there is a natural number p where $N = 2p + 3$.*

PROOF. By assumption, $N = 2a + 3b$ for some $b \geq 0$.

CASE 1: b is even. In this case, $b = 2 * m$ for some m, and so

$$\begin{aligned} N &= 2a + 3b \\ &= 2a + 3 * (2 * m) \quad \text{(since } b = 2m\text{)} \\ &= 2 * (a + 3m) + 3 * 0, \quad \text{(regrouping)} \end{aligned}$$

establishing that N is an even number.

CASE 2: b is odd. In this case, $b = 2q + 1$ for some q, the definition of an odd number. Thus

$$\begin{aligned} N &= 2a + 3b \\ &= 2a + 3 * (2q + 1) \quad \text{(since } b = 2q + 1\text{)} \\ &= (2a + 6p) + 3 * 1 \quad \text{(regrouping)} \\ &= 2(a + 3q) + 3 * 1, \end{aligned}$$

establishing the claim.

THEOREM: *Let $N \geq 5$ be a natural number. Then n can be written as the sum $2a + 3b$, where a and b are natural numbers.*

PROOF. The theorem is clearly true for $N = 5$. Suppose for contradiction there exists a natural number $N > 5$ that cannot be written as $N = 2a + 3b$ for any natural numbers a and b. Let N be the smallest such natural number (the inductive hypothesis).

Then $(N - 1)$ can be written as the required sum, i.e., there exist natural numbers a and b such that $(N - 1) = 2a + 3b$. By the Lemma, either $N - 1$ is even $(b = 0)$ or we may take $b = 1$.

CASE 1. $b = 0$. In this case, $N - 1 = 2a$, so $N = 2a + 1$. Since $N > 5$ it follows that $a > 2$, and thus that $(a - 1) > 1$. Thus,

$$\begin{aligned} N &= 2a + 1 \\ &= 2(a - 1 + 1) + 1 \\ &= 2(a - 1) + 3 \end{aligned}$$

contradicting N as the least integer for which the theorem fails.

CASE 2. $b = 1$. Via the lemma, $N - 1 = 2a + 3$ for some a. But then, adding one to both sides gives $N = 2a + 4 = 2 * (a + 2)$, contradicting the choice of N as the least number for which the theorem fails.