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## XII. Tests for Proportions: One Sample problems

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*Scenario:* You will have sample observations from an experimental population and census data from a control population. The observations will place the subject in categories and hence will be suitable for computations involving proportions. The research objectives will concentrate on one outcome which will be designated “success.” The “success” designation is assigned because this category is the one you are studying, not necessarily because it constitutes “success” in any conventional sense. Observations on the experimental group have been taken only once, after exposure to the experimental treatment.

✓ For the *Experimental Population* you will have only sample data:

	Parameters (unknown)	Sample Estimates (known)
	—	Number of successes $k$
Proportion	$p_1$	$\hat{p}$
sample size	—	$n$

The sample proportion  $\hat{p}$  is computed according to the formula:

$$\begin{aligned} \hat{p} &= \frac{\text{number of “successes”}}{\text{sample size}} \\ &= \frac{k}{n} \end{aligned}$$

✓ For the *Control Population* you will have census data:

	Parameter (known)
Proportion	$p_0$

✓ Research Objective: Determine if the observed relationship between the proportions observed in the experimental sample and the control population is due to

- the experimental treatment; or
- random chance inherent in sampling.

————— *Solution Template* —————

**Step 1.** Make a dictionary assigning values to each of the variables:

sample size	<b><i>n</i></b>
number of “successes”	<b><i>k</i></b>
sample proportion	$\hat{p} = \frac{k}{n}$
proportion in the control population	<b><i>p</i><sub>0</sub></b>
significance level	<b><i>α</i></b>

Make sure that you express  $\hat{p}$  and  $p_0$  as proportions and not as percentages. The ensuing formulae will not work with percentages. Also check to see that both

$$np_0 \geq 5$$

and

$$n(1 - p_0) \geq 5$$

a requirement for using this procedure.

**Step 2.** Write down the null and alternative hypotheses. The null hypothesis will always be:

$$H_0 : p_1 = p_0$$

while the alternative hypothesis will be one of the following:

$$H_A : p_1 < p_0 \quad (\text{a left tailed test})$$

$$H_A : p_1 > p_0 \quad (\text{a right tailed test})$$

$$H_A : p_1 \neq p_0 \quad (\text{a two tailed test})$$

(The reason for the terms right, left and two tailed tests is given in step 5.)

**Step 3.** Compute the value of the test statistic (*make sure that  $\hat{p}$  and  $p_0$  are expressed as proportions and not as percentages*):

$$\frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

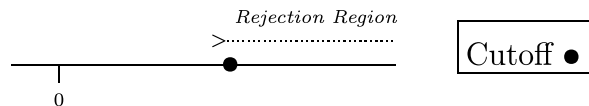
This test statistic is approximately normally distributed if both  $np_0$  and  $n(1 - p_0)$  are larger than five.

**Step 4.** Find the cutoff(s) using the table in Appendix B.

- Find the significance level in the first column in the table.
- Determine which kind of test you are running from the alternative hypothesis; find the corresponding column at the top of the table.
- The entry in the intersection of the above row and column is the cutoff corresponding to the test. If you are running a two tailed test, you will actually find two cutoffs.

**Step 5.** Decision rules (it often helps to plot the cutoff(s) and the value of the test statistic on a number line):

- First plot the cutoff(s) and the rejection region(s) on a number line.
  - If  $H_A : \mu_1 > \mu_0$ : This is called a *right-tailed* test since the rejection region is to the right of the cutoff.

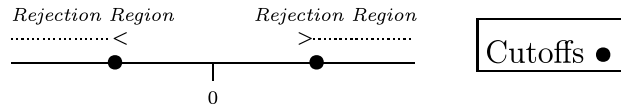


- If  $H_A : \mu_1 < \mu_0$ : This is called a *left-tailed* test since the rejection region is to the left of the cutoff. Don't forget that the cutoff will be negative.



- If  $H_A : \mu_1 \neq \mu_0$ : This is called a *two-tailed* test since there are two rejection regions: one to the left of the negative cutoff and one to the right of

the positive cutoff.



- Plot the value of the test statistic on the number line. Reject the null hypothesis if the value of the test statistic is in the rejection region. (For two tailed tests, reject  $H_0$  if the test statistic falls in *either* of the two rejection regions.)

Of course, if you *reject* the null hypothesis then you *accept* the alternative hypothesis. If you *do NOT reject the null hypothesis* then you *accept* the null hypothesis and *reject* the alternative hypothesis.

**End of Solution Template**

**Interpretation:** Hypothesis tests let you choose between accepting and rejecting the null hypothesis. No matter which decision you make you have possibly made an error. The following table shows exactly when those errors can occur.

	$H_0$ true	$H_A$ true
accept $H_0$	OK	Type II error
reject $H_0$	Type I error	OK

If your decision is to reject the null hypothesis, then you have possibly made a Type I error. The significance level  $\alpha$  is the largest chance of Type I error which you are willing to tolerate. The researcher is responsible for determining what this level should be, but there are some generally accepted standards for what various levels of  $\alpha$  should mean:

$\alpha > 5\%$	results are not statistically significant
$1\% < \alpha \leq 5\%$	results are statistically significant
$\alpha \leq 1\%$	results are highly significant

While the researcher is free to assign any value to  $\alpha$ , deviation from the above standards should be carefully justified.

If your decision is to accept the null hypothesis you have potentially made a Type II error. This kind of error is not generally controlled in hypothesis testing. This is

because hypothesis tests are designed to be conservative: we believe that the experimental treatment made no difference unless there is compelling evidence to the contrary. It also turns out to be much more difficult theoretically to compute the chances of Type II error.

**Hints on how to determine the alternative hypothesis.** Which form  $H_A$  takes is determined by your research objective. You can often guess which form is appropriate for  $H_A$  by looking at the relationship suggested by the data: suppose for example that  $\hat{p} < p_0$ . If  $H_A$  were  $p_1 > p_0$  there would be no reason to run a complicated statistical test since the data do not even appear to support the alternative. For this reason, if  $\hat{p} < p_0$  then  $H_A$  is probably  $p_1 < p_0$ . Similar reasoning applies if  $\hat{p} > p_0$ , suggesting  $H_A$  is probably  $p_1 > p_0$ .

Generally hypothesis tests involve an experimental treatment. The researcher performs the experiment because the researcher suspects that the treatment will increase or decrease the response measured by the proportion. For this reason, in most cases the researcher will use one-sided rather than two sided alternatives.

**Assumptions.** You must assume that both

$$np_0 \geq 5$$

and

$$n(1 - p_0) \geq 5.$$

**Example.** Sixty two percent of patients suffering their first heart attack will, on average, suffer another heart attack within five years. You have a sample of 500 patients who have suffered a first heart attack. In addition to the standard medical therapy, these patients will participate in group training session for the first month following discharge from the hospital; the training will focus on the life style changes needed to reduce the risk of further heart attacks. After five years, you discover that 293 of your original 500 patients have had a second heart attack. At the  $\alpha = 5\%$  significance level, is this sufficient evidence to recommend that all first-time heart attack patients receive this therapy?

**Solution.**

**Step 1.** We first need to identify the parameters. In this problem, we are interested

in studying those patients who have had a second heart attack within five years; thus, “success” becomes having a second heart attack since this is the focus of our study. Making a dictionary of variables:

sample size	$n$	500
number of “successes”	$k$	293
sample proportion	$\hat{p} = \frac{k}{n}$	$\frac{293}{500} = 0.586$
proportion in the control population	$p_0$	0.62
significance level	$\alpha$	5%

Since  $np_0 = 0.62(500) = 310$  and  $n(1 - p_0) = 500(.38) = 190$  are both well over 5, we can proceed. Also note that we have converted “62%” to the proportion “0.62.”

**Step 2.** Write down the null and alternative hypotheses. Since we are conjecturing that training reduces the risk of a second heart attack, these are:

$$H_0 : p_1 = 0.62$$

$$H_A : p_1 < 0.62$$

Note that we will not choose to incur the additional expense of training unless there is good reason to do so.

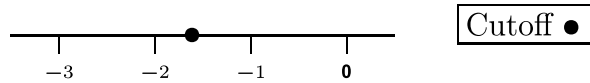
**Step 3.** Next we compute the test statistic:

$$\begin{aligned} \text{test statistic} &= \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \times (1 - p_0)}{n}}} \\ &= \frac{0.586 - 0.62}{\sqrt{\frac{0.62 \times (1 - 0.62)}{500}}} \\ &= \frac{-0.034}{\sqrt{\frac{0.2356}{500}}} \\ &= \frac{-0.034}{0.0217} \\ &= -1.57 \end{aligned}$$

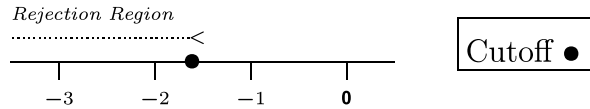
and hence the test statistic is -1.57.

**Step 4.** From the table in Appendix B, the cutoff for a left-tailed 5% test is -1.644.

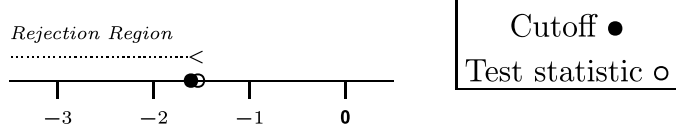
**Step 5.** To help make the decision whether or not to reject the null hypothesis, first draw a number line and plot the cutoff from step 4.



Next find the rejection region.



Finally, plot the value of the test statistic.



Since the test statistic is *not* in the rejection region, we do *NOT* reject the null hypothesis. If we do not reject the null hypothesis, we must accept it. This means that we have not found convincing evidence that the life style counselling reduces the risk of a second heart attack. **I**

*Question 1.* Suppose that you had specified an  $\alpha$  of 0.10; would you then have rejected or accepted  $H_0$ ? Which significance level is more appropriate in this case?

*Question 2.* Are there other uncontrolled factors which may have influenced the rate at which second heart attacks occur?

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## Problems

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1. In a sample of size 1864 there were 112 successes. Test the null hypothesis

$$H_0 : p = 0.02$$

against the alternative hypothesis

$$H_A : p > 0.02$$

using a significance level of 5%.

2. In a sample of size 400 there were 36 successes. Test the null hypothesis

$$H_0 : p = 0.09$$

against the alternative hypothesis

$$H_A : p > 0.09$$

using a significance level of 5%.

3. In a sample of size 2,000 there were 180 successes. Test the null hypothesis

$$H_0 : p = 0.09$$

against the alternative hypothesis

$$H_A : p > 0.09$$

using a significance level of 5%. Why do you get a different result in problems two and three even though  $\hat{p}$ ,  $p_0$  and the significance level are all the same?

4. A soda company claims that its new product, BlastOff Cola, has a distinctive taste which consumers will prefer. To test this conjecture, a sample of 900 consumers were given a taste test. They were presented with unlabeled samples of Coca-Cola, Pepsi and BlastOff Cola. The researchers discovered that 326 of the consumers preferred BlastOff Cola, 300 preferred Coca Cola and 274 preferred Pepsi. The company claims in advertising that this shows that “consumers prefer BlastOff.”

- (i) You suspect that consumers really can't tell the difference between the three colas. Use BlastOff's data to test your hypothesis at the 5% level. (Hint: if consumers can't tell the difference, then each cola is equally likely to be selected as “best” by any given consumer, i.e., the proportion preferring BlastOff should be one third, the proportion preferring Coca Cola should be one third and the proportion preferring Pepsi should be one third.)
- (ii) In BlastOff's experiment, the last cola sampled by all 900 consumers was BlastOff. Describe a better protocol for administering the taste test.

5. A researcher is interested in learning if sexual orientation is due to environmental factors or to hereditary factors. To learn about this, the researcher randomly sampled 283 identical male twins. The twins were selected from a population of twins which had all been raised separately (controlling for environment) and at least one member of the pair was homosexual. In this group, 66 of the pairs were both homosexual. Using the same definition of homosexuality, it is known that 8.9% of the general male population is homosexual. Have the researchers gathered convincing evidence that heredity influences sexual orientation? Use a significance level of 1%.



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**Video Assignment.**

View the following program(s) from the series *Against All Odds*:

Program	Title
23	<i>Inference for Proportions</i>